The Log-Rank Conjecture: New Equivalent Formulations

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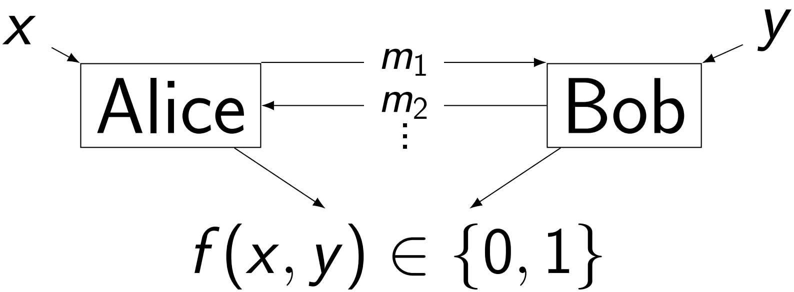
Outline

- What is the Log-Rank Conjecture?
- Equiv. formulation 1: signed rectangle rank
- Equiv. formulation 2: cross-intersecting sets

Communication Complexity



How much information must be exchanged to compute f?

Communication as matrices

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 & \downarrow & & & & & & & \downarrow & & & & & & & & \\
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x \rightarrow & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
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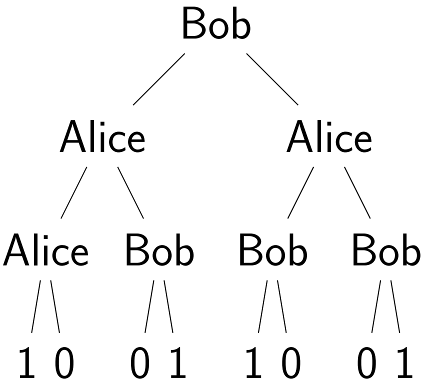
\hline
\end{array}
$$

Communication as matrices



$$
\begin{array}{|ccccccccc|ccccc|cc|}
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0&0&0&0&0&1&1&1&1&0&0&1&1&1&1&1\\
\hline
\end{array}
$$

$\#$ all-ones rectangles $\leq$ $\#$ of protocol leaves $\leq 2^{D^{cc}}$

Connection between communication and rank

$$\begin{matrix}1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\1&1&1&1&1&1&1&1&1&0&0&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&0&0&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&0&0&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&0&0&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&1&1&1&1&0&0&1&1&1&1&1\\0&0&0&0&0&1&1&1&1&0&0&1&1&1&1&1\\0&0&0&0&0&1&1&1&1&0&0&1&1&1&1&1\\0&0&0&0&0&1&1&1&1&0&0&1&1&1&1&1\end{matrix} = \begin{matrix}0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0\\0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0\\0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0\\0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0\end{matrix} + \begin{matrix}1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\end{matrix} + \begin{matrix}0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\end{matrix} + \begin{matrix}0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0\\0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0\\0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\end{matrix}$$

$$\mathrm{D^{cc}}(M) \geq \log \mathrm{rank}(M)$$

$$\mathrm{D^{cc}}(M) \geq \log \mathrm{binrk}(M) \geq \log \mathrm{rank}(M)$$

(Binary rank is also called *partition number*)

$$\mathrm{D^{cc}}(M) \leq \mathrm{poly} \log \mathrm{binrk}(M)$$

The Log-Rank Conjecture

Conjecture [LS88]: for all Boolean matrices,
$\mathrm{D}^{\mathrm{cc}}(M) \leq \operatorname{poly}\log \operatorname{rank}(M)$

Because D^{cc} and $\log \operatorname{binrk}$ are polynomially-related, this is equivalent to

Conjecture: for all Boolean matrices,
$\log \operatorname{binrk}(M) \leq \operatorname{poly}\log \operatorname{rank}(M)$

Why do we care?

- It would answer questions about the structure of low-rank matrices
- Hardness of approximation for D^{cc}:
  - D^{cc} NP-hard to compute [HIL25]
  - Binary rank NP-hard to compute [JR93]
  - Rank poly-time to compute (Gaussian elimination)
- Graph theory implications: chromatic number vs. adjacency matrix rank

How is rank different from binary rank?

- Rank is **algebraic**, binary rank is **combinatorial**
- Rank allows **cancellations**, binary rank is **monotone**

What happens when we relax only one of these?

Monotone but algebraic

Non-negative rank: How many *non-negative* rank-one matrices are needed to sum to M?

$$\log \operatorname{binrk}(M) \leq \operatorname{poly} \log \operatorname{nnr}(M)$$

Combinatorial but with cancellations

Definition: The signed rectangle rank of M, denoted $\mathrm{srr}(M)$, is the minimum s such that we can write

$$M = \sum_{i=1}^{s} \epsilon_i R_i$$

where R_i are binary rank-one matrices and $\epsilon_i \in \{-1, 1\}$.

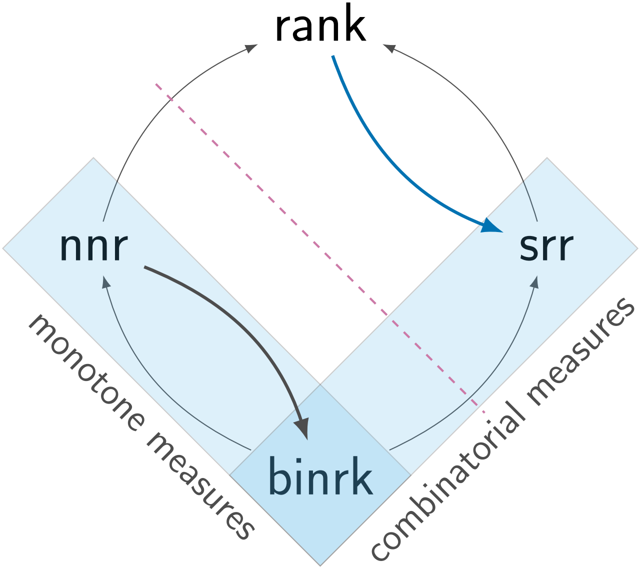
Main result

Theorem: For every Boolean matrix M of rank r,

$$\operatorname{srr}(M) \leq O(r \log r).$$

So the Log-Rank Conjecture is *equivalent* to:

Conjecture: for all Boolean matrices,
$\log \operatorname{binrk}(M) \leq \operatorname{poly}\log \operatorname{srr}(M)$


rank
nnr
srr
binrk
monotone measures
combinatorial measures

Cancellations vs. monotonicity

is the core of the Log-Rank Conjecture

Secondary result

We also prove that the Log-Rank Conjecture is equivalent to a conjecture in combinatorics.

Main result: proof idea

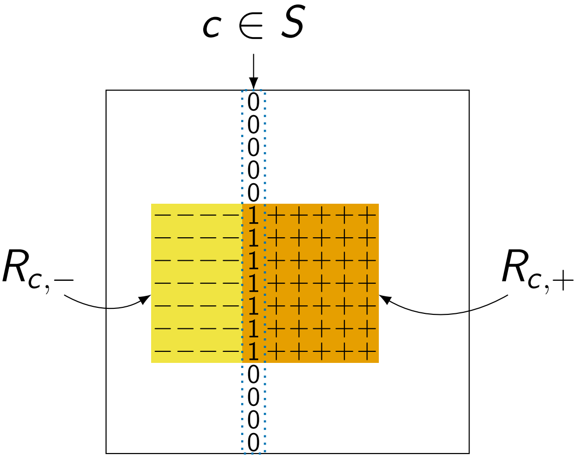
Theorem: For every Boolean matrix M of rank r,

$$\mathrm{srr}(M) \leq O(r \log r).$$

Idea: Find a set of columns S such that:

- $|S| \leq O(r \log r)$
- Every column in M is a ± 1 linear combination of columns from S

Main result: proof idea



$$\mathrm{srr}(M) \leq 2|S| \leq O(r \log r)$$

Main result: proof idea

Idea: Find an <u>independent</u> set of columns S such that:

- $|S| \leq O(r \log r)$
  - **Claim 1**
- Every column in M is a ± 1 linear combination of columns from S
  - **Claim 2**

Main result: independent sets of columns

Definition: A set of columns S is *independent* if all subsets of S have different sums.

Weaker than linear independence.

$$\begin{pmatrix}1\\0\\1\end{pmatrix} \begin{pmatrix}0\\1\\1\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 1 & 1 & 2 \end{bmatrix}$$

Main result: outline

Claim 1: If S is an independent set of M, then $|S| \leq O(r \log r)$.

Claim 2: If S is a maximal independent set of M, then every column in M is a ± 1 linear combination of columns from S.

Main result: Claim 1

Claim 1: If S is an independent set of M, then $|S| \leq O(r \log r)$.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 1 & 1 & 2 \end{bmatrix}$$

- $\#$ columns $= \#$ unique columns $= 2^{|S|}$
- rank $\leq r$
- entries $\in \{0, \ldots, |S|\}$

Main result: Claim 1

Claim 1: If S is an independent set of M, then $|S| \leq O(r \log r)$.

$$\text{rank} \leq r \qquad \text{entries} \in \{0, \ldots, |S|\}$$

- r rows span the matrix
- Columns uniquely identified by r entries
- At most $(|S| + 1)^r$ unique columns

Main result: Claim 1

Claim 1: If S is an independent set of M, then $|S| \leq O(r \log r)$.

$$(|S| + 1)^r \leq \# \text{ unique columns} = 2^{|S|}$$

Solving for $|S|$ gives $|S| \leq O(r \log r)$.

Main result: Claim 2

Claim 2: If S is a maximal independent set of M, then every column in M is a ± 1 linear combination of columns from S.

For any column c of M:

- If $c \in S$, we're done :)
- Else, $S \cup \{c\}$ not independent
  - $\sum A = \sum B + c$
  - $c = \sum A - \sum B$
  - done here too :)

Main result

Theorem: For every Boolean matrix M of rank r,

$$\mathrm{srr}(M) \leq O(r \log r).$$

Claim 1: If S is an independent set of M, then $|S| \leq O(r \log r)$.

Claim 2: If S is a maximal independent set of M, then every column in M is a ± 1 linear combination of columns from S.

Cross-intersecting set families

A pair of set families $\mathcal{S}, \mathcal{T} \subseteq 2^{[d]}$ are $\{a, b\}$-*cross-intersecting* if

$$\forall (S, T) \in \mathcal{S} \times \mathcal{T}, |S \cap T| \in \{a, b\}.$$

Cross-intersecting set families conjecture

Conjecture [L21,SS22]: For any a, b and any $\{a, b\}$-cross-intersecting families $\mathcal{S}, \mathcal{T} \subseteq 2^{[d]}$, there exist sub-families $\mathcal{A} \subseteq \mathcal{S}, \mathcal{B} \subseteq \mathcal{T}$ such that

- Every pair $(A, B) \in \mathcal{A} \times \mathcal{B}$ has the same intersection size,
- $|\mathcal{A}| \cdot |\mathcal{B}| \geq 2^{\text{poly} \log d} \cdot |\mathcal{S}| \cdot |\mathcal{T}|$.

A similar conjecture

Conjecture [NW95]: Any Boolean matrix M has an all-zeroes or all-ones rectangle of density $2^{\operatorname{poly}\log \operatorname{rank}(M)}$.

Theorem [NW95]: The above conjecture is equivalent to the log-rank conjecture.

Our result on the CIS conjecture

Using the Main Theorem, we prove:

Theorem: The cross-intersecting set conjecture implies the Log-Rank Conjecture.

(Therefore, the conjectures are equivalent!)

Concluding thoughts

- Cancellations vs. monotonicity: "right way" to think about the LRC?
- Pessimist's view: CIS conjecture seems hard
- Optimist's view: new reason to care about CIS conjecture
- Left out of talk: methods naturally extend to tensors and tensor rank, related to NIH LRC
- Open problem: improve $O(r \log r)$ to $O(r)$? Our method doesn't work for this [Lin65]